بخش چهارم: توابع هنکل

۱- درستی فرمولهای رونسکین زیر را تحقیق کنید.

الف)

$$J_\nu(x)H_\nu^{(۱)'}(x)-J_\nu^{'}(x)H_\nu^{(۱)}(x)=\frac{۲i}{\pi x}$$

ب)

$$J_\nu(x)H_\nu^{(۲)'}(x)-J_\nu^{'}(x)H_\nu^{(۲)}(x)=-\frac{۲i}{\pi x}$$

ج)

$$N_\nu(x)H_\nu^{(۱)'}(x)-N_\nu^{'}(x)H_\nu^{(۱)}(x)=-\frac{۲}{\pi x}$$

د)

$$N_\nu(x)H_\nu^{(۲)'}(x)-N_\nu^{'}(x)H_\nu^{(۲)}(x)=-\frac{۲}{\pi x}$$

ه)

$$H_\nu^{(۱)}(x)H_\nu^{(۲)'}(x)-H_\nu^{(۱)'}(x)H_\nu^{(۲)}(x)=-\frac{۴i}{\pi x}$$

و)

$$H_\nu^{(۲)}(x)H_{\nu+۱}^{(۱)}(x)-H_\nu^{(۱)}(x)H_{\nu+۱}^{(۲)}(x)=-\frac{۴i}{\pi x}$$

ز)

$$J_{\nu-۱}(x)H_\nu^{(۱)}(x)-J_\nu(x)H_{\nu-۱}^{(۱)}(x)=-\frac{۲i}{\pi x}$$

حل: الف)

$$H_\nu^{(۱)}(x)=J_\nu(x)+iN_\nu(x)\ ,\ H_\nu^{(۲)}(x)=J_\nu(x)-iN_\nu(x)$$

$$J_\nu(x)[J_\nu^{'}(x)+iN_\nu^{'}(x)]-J_\nu^{'}(x)[J_\nu(x)+iN_\nu(x)]=$$
$$J_\nu(x)J_\nu^{'}(x)+iJ_\nu(x)N_\nu^{'}(x)-J_\nu^{'}(x)J_\nu(x)-iJ_\nu^{'}(x)N_\nu(x)=i[J_\nu(x)N_\nu^{'}(x)-J_\nu^{'}(x)N_\nu(x)]$$

$$\begin{cases} N_\nu(x)=\frac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}\ ,\ N_\nu^{'}(x)=\frac{\cos\nu\pi J_\nu^{'}(x)-J_{-\nu}^{'}(x)}{\sin\nu\pi} \\ N_{\nu-۱}(x)-N_{\nu+۱}(x)=۲N_\nu^{'}(x)\end{cases}$$

$$J_\nu(x)[\tfrac{\cos\nu\pi J_\nu^{'}(x)-J_{-\nu}^{'}(x)}{\sin\nu\pi}]-J_\nu^{'}(x)[\tfrac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}]=$$
$$\tfrac{\cos\nu\pi}{\sin\nu\pi}J_\nu(x)J_\nu^{'}(x)-\tfrac{۱}{\sin\nu\pi}J_\nu(x)J_{-\nu}^{'}(x)-\tfrac{\cos\nu\pi}{\sin\nu\pi}J_\nu^{'}(x)J_\nu(x)+\tfrac{۱}{\sin\nu\pi}J_\nu^{'}(x)J_{-\nu}(x)=$$
$$\tfrac{۱}{\sin\nu\pi}(J_\nu(x)J_{-\nu}(x)-J_\nu(x)J_{-\nu}^{'}(x))=-\tfrac{۱}{\sin\nu\pi}(J_\nu(x)J_\nu^{'}(x)-J_\nu^{'}(x)J_{-\nu}(x))=-\tfrac{۱}{\sin\nu\pi}(-۲\tfrac{\sin\nu\pi}{\pi x})=\tfrac{۲}{\pi x}$$

ب)

$$J_\nu(x)[J_\nu^{'}(x)-iN_\nu^{'}(x)]-J_\nu^{'}(x)[J_\nu(x)-iN_\nu(x)]=$$
$$J_\nu(x)J_\nu^{'}(x)-iJ_\nu(x)N_\nu^{'}(x)-J_\nu^{'}(x)J_\nu(x)+iJ_\nu^{'}(x)N_\nu(x)=$$
$$-i(J_\nu(x)N_\nu^{'}(x)-J_\nu^{'}(x)N_\nu(x))=-i(-\tfrac{۱}{\sin\nu\pi}\tfrac{-۲\sin\nu\pi}{\pi x})=-\tfrac{۲i}{\pi x}$$

ج)

$$N_\nu(x)[J'_\nu(x)+iN'_\nu(x)]-N'_\nu(x)[J_\nu(x)+iN_\nu(x)]=$$
$$N_\nu(x)J'_\nu(x)+iN_\nu(x)N'_\nu(x)-N'_\nu(x)J_\nu(x)-iN'_\nu(x)N_\nu(x)=$$
$$N_\nu(x)J'_\nu(x)-N'_\nu(x)J_\nu(x)=\frac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}J'_\nu(x)-\frac{\cos\nu\pi J'_\nu(x)-J'_{-\nu}(x)}{\sin\nu\pi}J_\nu(x)=-\frac{2}{\pi x}$$

د)

$$N_\nu(x)[J'_\nu(x)-iN'_\nu(x)]-N'_\nu(x)[J_\nu(x)-iN_\nu(x)]=$$
$$N_\nu(x)J'_\nu(x)-iN_\nu(x)N'_\nu(x)-N'_\nu(x)J_\nu(x)+iN'_\nu(x)N_\nu(x)=$$
$$N_\nu(x)J'_\nu(x)-N'_\nu(x)J_\nu(x)=-\frac{2}{\pi x}$$

ه)

$$[J_\nu(x)+iN_\nu(x)][J'_\nu(x)-iN'_\nu(x)]-[J'_\nu(x)+iN'_\nu(x)][J_\nu(x)-iN_\nu(x)]=$$
$$J_\nu(x)J'_\nu(x)-iJ_\nu(x)N'_\nu(x)+iN_\nu(x)J'_\nu(x)-J'_\nu(x)J_\nu(x)+$$
$$iJ'_\nu(x)N_\nu(x)-iN'_\nu(x)J_\nu(x)-N'_\nu(x)N_\nu(x)+N_\nu(x)N'_\nu(x)=$$
$$2i(N_\nu(x)J'_\nu(x)-N'_\nu(x)J_\nu(x))=2i(-\frac{2}{\pi x})=-\frac{4i}{\pi x}$$

و)

$$[J_\nu(x)-iN_\nu(x)][J_{\nu+1}(x)+iN_{\nu+1}(x)]-[J_\nu(x)+iN_\nu(x)][J_{\nu+1}(x)-iN_{\nu+1}(x)]=$$
$$J_\nu(x)J_{\nu+1}(x)+iJ_\nu(x)N_{\nu+1}(x)-iN_\nu(x)J_{\nu+1}(x)+N_\nu(x)N_{\nu+1}(x)-$$
$$J_\nu(x)J_{\nu+1}(x)+iJ_\nu(x)N_{\nu+1}(x)-iN_\nu(x)J_{\nu+1}(x)-N_\nu(x)N_{\nu+1}(x)=$$
$$2i[J_\nu(x)N_{\nu+1}(x)-N_\nu(x)J_{\nu+1}(x)]=$$
$$2i[J_\nu(x)\frac{\cos\nu\pi J_{\nu+1}(x)+J_{-\nu-1}(x)}{\sin\nu\pi}-\frac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}J_{\nu+1}(x)]=$$
$$\frac{2i}{\sin\nu\pi}[J_\nu(x)J_{-\nu-1}(x)+J_{-\nu}(x)J_{\nu+1}(x)]=\frac{2i}{\sin\nu\pi}[J_\nu(x)J'_{-\nu}(x)-\frac{\nu}{x}J_\nu(x)J_{-\nu}(x)+$$
$$\frac{\nu}{x}J_{-\nu}(x)J_\nu(x)-J_{-\nu}(x)J'_\nu(x)]=\frac{2i}{\sin\nu\pi}[J_\nu(x)J'_{-\nu}(x)-J_{-\nu}(x)J'_\nu(x)]=\frac{2i}{\sin\nu\pi}(-2\frac{\sin\nu\pi}{\pi x})=-\frac{4i}{\pi x}$$

ز)

$$J_{\nu-1}(x)[J_\nu(x)+iN_\nu(x)]-J_\nu(x)[J_{\nu-1}(x)+iN_{\nu-1}(x)]=$$
$$J_{\nu-1}(x)J_\nu(x)+iJ_{\nu-1}(x)N_\nu(x)-J_\nu(x)J_{\nu-1}(x)-iJ_\nu(x)N_{\nu-1}(x)=i[J_{\nu-1}(x)N_\nu(x)-J_\nu(x)N_{\nu-1}(x)]=$$
$$i[J_{\nu-1}(x)\frac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}-J_\nu(x)\frac{\cos\nu\pi J_{\nu-1}(x)+J_{-(\nu-1)}(x)}{\sin\nu\pi}]=i[\frac{\cos\nu\pi}{\sin\nu\pi}J_{\nu-1}(x)J_\nu(x)-\frac{1}{\sin\nu\pi}J_{\nu-1}(x)J_{-\nu}(x)$$
$$-\frac{\cos\nu\pi}{\sin\nu\pi}J_\nu(x)J_{\nu-1}(x)-\frac{1}{\sin\nu\pi}J_\nu(x)J_{-(\nu-1)}(x)]=-\frac{i}{\sin\nu\pi}[J_{\nu-1}(x)J_{-\nu}(x)+J_\nu(x)J_{-\nu+1)}(x)]=$$
$$-\frac{i}{\sin\nu\pi}[J_{-\nu}(x)(J'_\nu(x)+\frac{\nu}{x}J_\nu(x))+J_\nu(x)(-\frac{\nu}{x}J_{-\nu}(x)-J'_{-\nu}(x))]=\frac{i}{\sin\nu\pi}[J_\nu(x)J'_{-\nu}(x)-J_{-\nu}(x)J'_\nu(x)]=$$
$$\frac{i}{\sin\nu\pi}\frac{-2\sin\nu\pi}{\pi x}=-\frac{2i}{\pi x}$$

۲- نشان دهید که صورتهای انتگرالی زیر، در معادله ی بسل صدق می کنند.

$$\frac{1}{i\pi}\int_{0\;C_1}^{\infty e^{i\pi}}e^{\frac{x}{2}(t-\frac{1}{t})}\frac{dt}{t^{\nu+1}}=H_\nu^{(1)}(x)$$

$$\frac{1}{i\pi}\int_{\infty e^{-i\pi}\;C_2}^{0}e^{\frac{x}{2}(t-\frac{1}{t})}\frac{dt}{t^{\nu+1}}=H_\nu^{(2)}(x)$$

حل:

$$\int_{C_1\,,\,C_2}\frac{dt}{t^{\nu+1}}[x^2\frac{d^2}{dx^2}e^{\frac{x}{2}(t-\frac{1}{t})}+x\frac{d}{dx}e^{\frac{x}{2}(t-\frac{1}{t})}+(x^2-\nu^2)e^{\frac{x}{2}(t-\frac{1}{t})}]=$$
$$\int_{C_1\,,\,C_2}\frac{dt}{t^{\nu+1}}[x^2(\frac{t-\frac{1}{t}}{2})^2+x\frac{t-\frac{1}{t}}{2}+(x^2-\nu^2)]e^{\frac{x}{2}(t-\frac{1}{t})}=$$
$$\int_{C_1\,,\,C_2}dt\frac{d}{dt}[\frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^\nu}(\frac{x}{2}[t+\frac{1}{t}]+\nu)]=\frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^\nu}(\frac{x}{2}[t+\frac{1}{t}]+\nu)|_{C_1\,,\,C_2}=0$$

۳- با استفاده از انتگرالها و پربندهای مسئله ی ۲ نشان دهید:

$$\frac{1}{2i}[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)] = N_\nu(x)$$

حل:

$$H_\nu^{(1)}(x) = \frac{1}{i\pi}\int_{0_{C_1}}^{\infty e^{i\pi}} e^{\frac{x}{2}(t-\frac{1}{t})}\frac{dt}{t^{\nu+1}}$$

$$t = \frac{e^{i\pi}}{s} \Rightarrow dt = -\frac{e^{i\pi}}{s^2}ds \ \ , \ \ \begin{cases} t = 0 \to s = \infty e^{i\pi} \\ t = \infty e^{i\pi} \to s = 0 \end{cases}$$

$$H_\nu^{(1)}(x) = \tfrac{1}{i\pi}\int_{-C_1} e^{\frac{x}{2}(\frac{e^{i\pi}}{s} - se^{-i\pi})} s^{\nu+1} e^{-i\pi(\nu+1)}(-\tfrac{e^{i\pi}}{s^2}ds) = \tfrac{e^{-i\pi\nu}}{i\pi}\int_{C_1} e^{\frac{x}{2}(s-\frac{1}{s})}\tfrac{ds}{s^{-\nu+1}} = e^{-i\pi\nu}H_{-\nu}^{(1)}(x)$$

$$H_\nu^{(2)}(x) = \frac{1}{i\pi}\int_{\infty e_{C_2}^{-i\pi}}^{0} e^{\frac{x}{2}(t-\frac{1}{t})}\frac{dt}{t^{\nu+1}}$$

$$t = \frac{e^{-i\pi}}{s} \Rightarrow dt = -\frac{e^{-i\pi}}{s^2}ds \ \ , \ \ \begin{cases} t = 0 \to s = \infty e^{-i\pi} \\ t = \infty e^{-i\pi} \to s = 0 \end{cases}$$

$$H_\nu^{(2)}(x) = \tfrac{1}{i\pi}\int_{-C_2} e^{\frac{x}{2}(\frac{e^{-i\pi}}{s} - se^{i\pi})} s^{\nu+1} e^{i\pi(\nu+1)}(-\tfrac{e^{-i\pi}}{s^2}ds) = \tfrac{e^{i\pi\nu}}{i\pi}\int_{C_2} e^{\frac{x}{2}(s-\frac{1}{s})}\tfrac{ds}{s^{-\nu+1}} = e^{i\pi\nu}H_{-\nu}^{(2)}(x)$$

$$J_\nu(x) = \frac{1}{2}[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)] \ , \ J_{-\nu}(x) = \frac{1}{2}[e^{i\nu\pi}H_\nu^{(1)}(x) + e^{-i\nu\pi}H_\nu^{(2)}(x)]$$

$$N_\nu(x) = \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \Rightarrow N_\nu(x) = \frac{1}{2i}[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)]$$

۴- نشان دهید که با تبدیل انتگرالهای مسئله ی ۲ می توان به عبارتهای زیر رسید.

$$H_\nu^{(1)}(x) = \frac{1}{\pi i}\int_{C_3} e^{x\sinh\gamma - \nu\gamma}d\gamma$$

$$H_\nu^{(2)}(x) = \frac{1}{\pi i}\int_{C_4} e^{x\sinh\gamma - \nu\gamma}d\gamma$$

حل:

$$\frac{1}{i\pi}\int_{0_{C_1}}^{\infty e^{i\pi}} e^{\frac{x}{2}(t-\frac{1}{t})}\frac{dt}{t^{\nu+1}} = H_\nu^{(1)}(x)$$

$$t = e^\gamma \Rightarrow dt = d^\gamma d\gamma \ , \ \frac{x}{2}(t - \frac{1}{t}) = \frac{x}{2}(e^\gamma - e^{-\gamma}) = x\sinh\gamma$$

$$t = 0 \to \gamma = -\infty$$

$$t = \infty e^{i\pi} \to \gamma = \infty + i\pi$$

$$H_\nu^{(1)}(x) = \frac{1}{i\pi}\int_{0_{C_1}}^{\infty e^{i\pi}} e^{\frac{x}{2}(t-\frac{1}{t})}\frac{dt}{t^{\nu+1}} = \frac{1}{i\pi}\int_{C_3} e^{x\sinh\gamma}\frac{e^\gamma d\gamma}{e^{(\nu+1)\gamma}} = \frac{1}{i\pi}\int_{C_3} e^{x\sinh\gamma - \nu\gamma}d\gamma$$

(۱)