The KdV Equation and Solitons

Introduction

The Korteweg-de Vries (KdV) equation, formulated in 1895 by Korteweg and de Vries, models water waves. It contrasts sharply to the Burgers equation because it introduces no dissipation and the waves travel seemingly forever. In 1965, Zabusky and Kruskal named such waves *solitons*.

The KdV equation with boundary conditions and initial value for this model is formulated as

$$\begin{aligned} u_t + u_{xxx} &= 6uu_x \qquad \text{in } \Omega = [-8, 8] \\ u(-8, t) &= u(8, t), \qquad \text{periodic} \\ u(x, 0) &= -6\,\mathrm{sech}^2(x) \end{aligned}$$

The equation models the steepening and dispersion of wavefronts but does not support a train of simple harmonic waves. Such trains comprise the wavecrests normally associated with the ocean: simply a momentary constructive interference of contributing waves moving at different speeds. However, the equation does support solitons, single “humps” that travel without changing shape or speed for unexpectedly long distances.

Indeed, Perry and Schimke (Ref. 2) concluded from shipboard oceanographic measurements that bands of choppy water in the Andaman Sea, which lies east of the Bay of Bengal and west of Burma (Union of Myanmar) and Thailand, are associated with large-amplitude oceanic internal waves. Satellite images have since clarified that these waves originate on shallow banks on a layer between warm and cool water. Further, Osborne and Burch (Ref. 1) analyzed oceanographic data in an effort to assess the forces of underwater current fluctuations associated with such waves on offshore drilling rigs. They concluded that the visually observed roughness bands are caused by internal solitons that follow the KdV equation (Ref. 3).

A more recent development is the application of the KdV equation to another type of waves—light waves. Today solitons have their primary practical application in optical fibers. Specifically, a fiber’s linear dispersion properties level out a wave while the nonlinear properties give a focusing effect. The result is a very stable, long-lived pulse (Ref. 3). It is amazing that researchers have discovered a formula for such waves:

$$u = \frac{v}{\left[2\cosh^2\left(\frac{1}{2}\sqrt{v}\right)(x - vt - f)\right]}$$

This equation says that the pulse speed is what determines the pulse amplitude and the pulse width. The following simulation illustrates this effect. An initial pulse, which does not conform to the formula, immediately breaks down into two pulses of different amplitudes and speeds. The two new pulses follow the formula and thus can travel forever. While the formula does not reveal how solitons interact, the simulation shows that they can collide and reappear, seemingly unchanged, just as linear waves do, another counterintuitive observation that is difficult to observe without predictions by computing.

Model Definition

In the model, the term uu_x describes the focusing of a wave and u_{xxx} refers to its dispersion. The balancing of these two terms permits waves to travel with their shape unchanged.

Because COMSOL Multiphysics does not evaluate third derivatives directly, you rewrite the original equation above as a system of two variables to solve it:

$$\begin{aligned} u_{1t} + u_{2x} &= 6u_1u_{1x} \\ u_{1xx} &= u_2 \end{aligned}$$

Using the General Form PDE interface, you need to define two dependent variables, `u1` and `u2`, and identify the d_a, Γ, and F coefficients in the following equation:

$$d_a\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \Gamma = F$$

- Only the first equation has a time derivative, and it is with respect to u_1, so only $d_a(1,1)$ is 1; the other three components are zero.
- The divergence in this model is a space derivative with respect to x. This means that the Γ component from the first equation is u_2, which you type as `u2`. The Γ component from the second equation is u_{1x}, which you express using COMSOL Multiphysics syntax as `u1x`.
- The F term components are the right-hand side of the equations: F_1 is $6u_1u_{1x}$ (type `6*u1*u1x`), and F_2 is u_2 (type `u2`).

The initial condition for u_1 uses a hyperbolic cosine function to provide an interesting wave form to start with. For u_2, you must provide the second space derivative of this function to provide consistent initial conditions.

The boundary conditions are periodic boundary conditions: the solution at one end is always identical to the one at the other end of the domain.

Results

The following plot shows how solitons collide and reappear with their shape intact.

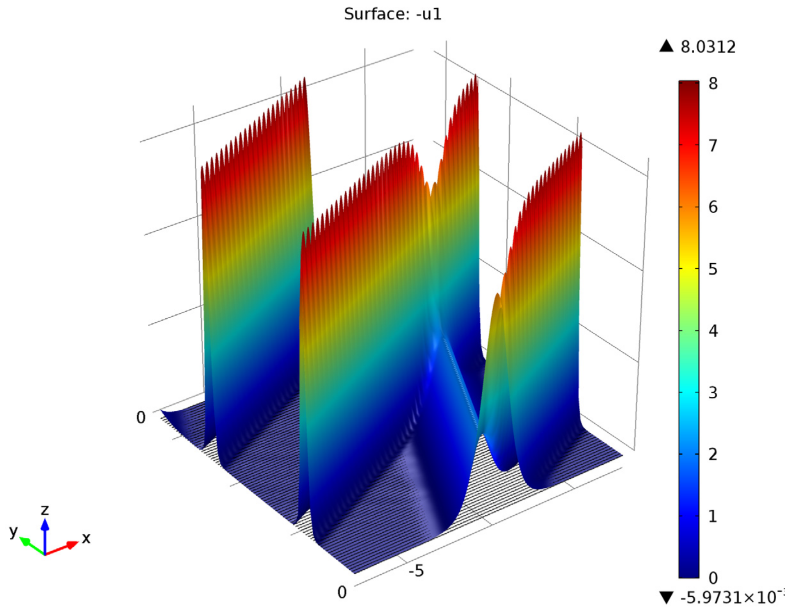

Figure 1: Solution visualizing a soliton collision.

References

1. A.R. Osborne and T.L. Burch, “Internal Solitons in the Andaman Sea,” *Science*, vol. 208, no. 4443, pp. 451–460, 1980.

2. R.B. Perry and G.R. Schimke, “Large-Amplitude Internal Waves Observed off the Northwest Coast of Sumatra,” *J. Geophys. Res.*, vol. 70, no. 10, pp. 2319–2324, 1965.

3. G. Strang, *Applied Mathematics*, Wellesley-Cambridge, 1986.

Model Library path: `COMSOL_Multiphysics/Equation-Based_Models/kdv_equation`

Modeling Instructions

MODEL WIZARD

1. Go to the **Model Wizard** window.
2. Click the **1D** button.
3. Click **Next**.
4. In the **Add physics** tree, select **Mathematics>PDE Interfaces>General Form PDE (g)**.
5. Click **Add Selected**.
6. Find the **Dependent variables** subsection. In the **Number of dependent variables** edit field, type `2`.
7. Click **Next**.
8. Find the **Studies** subsection. In the tree, select **Preset Studies>Time Dependent**.
9. Click **Finish**.

ROOT

1. In the **Model Builder** window, click the root node.
2. In the **Root** settings window, locate the **Unit System** section.
3. From the **Unit system** list, choose **None**.

   Keeping track of units is not important in this model; by turning off unit support, you avoid the need to specify dimensions for equation coefficients and coordinates to get rid of unit warnings.

GEOMETRY 1

Interval 1

1. In the **Model Builder** window, under **Model 1** right-click **Geometry 1** and choose **Interval**.
2. In the **Interval** settings window, locate the **Interval** section.
3. In the **Left endpoint** edit field, type `-8`.
4. In the **Right endpoint** edit field, type `8`.

GENERAL FORM PDE

Periodic Condition 1

1 In the **Model Builder** window, under **Model 1** right-click **General Form PDE** and choose **Periodic Condition**.
2 In the **Periodic Condition** settings window, locate the **Boundary Selection** section.
3 From the **Selection** list, choose **All boundaries**.

General Form PDE 1

1 In the **Model Builder** window, under **Model 1>General Form PDE** click **General Form PDE 1**.
2 In the **General Form PDE** settings window, locate the **Conservative Flux** section.
3 In the Γ edit-field array, type `u2` on the first row.
4 In the Γ edit-field array, type `u1x` on the 2nd row.
5 Locate the **Source Term** section. In the f edit-field array, type `6*u1*u1x` on the first row.
6 In the f edit-field array, type `u2` on the 2nd row.
7 Locate the **Damping or Mass Coefficient** section. In the d_a edit-field array, type `0` in the 2nd column of the 2nd row.

Initial Values 1

1 In the **Model Builder** window, under **Model 1>General Form PDE** click **Initial Values 1**.
2 In the **Initial Values** settings window, locate the **Initial Values** section.
3 In the $u1$ edit field, type `-6*sech(x)^2`.
4 In the $u2$ edit field, type `-24*sech(x)^2*tanh(x)^2+12*sech(x)^2*(1-tanh(x)^2)`.

MESH 1

In the **Model Builder** window, under **Model 1** right-click **Mesh 1** and choose **Edge**.

Size

1 In the **Model Builder** window, under **Model 1>Mesh 1** click **Size**.
2 In the **Size** settings window, locate the **Element Size Parameters** section.
3 In the **Maximum element size** edit field, type `0.1`.
4 Click the **Build All** button.

STUDY 1

Step 1: Time Dependent

1 In the **Model Builder** window, expand the **Study 1** node, then click **Step 1: Time Dependent**.
2 In the **Time Dependent** settings window, locate the **Study Settings** section.
3 In the **Times** edit field, type `range(0,0.025,2)`.
4 Select the **Relative tolerance** check box.
5 In the associated edit field, type `1e-4`.
6 In the **Model Builder** window, right-click **Study 1** and choose **Show Default Solver**.
7 Expand the **Study 1>Solver Configurations** node.

Solver 1

1 In the **Model Builder** window, expand the **Study 1>Solver Configurations>Solver 1** node, then click **Time-Dependent Solver 1**.
2 In the **Time-Dependent Solver** settings window, click to expand the **Time Stepping** section.
3 From the **Method** list, choose **Generalized alpha**.

   The Generalized-alpha time stepper is well suited for wave problems. For an accurate solution, use tighter tolerance settings.

4 Click to expand the **Absolute Tolerance** section. In the **Tolerance** edit field, type `1e-5`.
5 In the **Model Builder** window, right-click **Study 1** and choose **Compute**.

RESULTS

1D Plot Group 1

1 In the **Model Builder** window, under **Results** click **1D Plot Group 1**.
2 In the **1D Plot Group** settings window, locate the **Data** section.
3 From the **Time selection** list, choose **From list**.
4 In the **Times** list, select **0.25**.
5 In the **Model Builder** window, expand the **1D Plot Group 1** node, then click **Line Graph 1**.
6 In the **Line Graph** settings window, locate the **y-Axis Data** section.
7 In the **Expression** edit field, type `-u1`.
8 Click the **Plot** button.

Data Sets

In the **Model Builder** window, under **Results** right-click **Data Sets** and choose **More Data Sets>Parametric Extrusion 1D**.

2D Plot Group 2

1 Right-click **Results** and choose **2D Plot Group**.
2 In the **Model Builder** window, under **Results** right-click **2D Plot Group 2** and choose **Surface**.
3 In the **Surface** settings window, locate the **Expression** section.
4 In the **Expression** edit field, type `-u1`.
5 Right-click **Results>2D Plot Group 2>Surface 1** and choose **Height Expression**.
6 Click the **Zoom Extents** button on the Graphics toolbar.