

Real Analysis

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1 Absolute Value and Distance in \mathbb{R} .

Definition 1.1. Let $a \in \mathbb{R}$. Then the **absolute value** of a is defined by:

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Note. Insert image.

Definition 1.2. Let $a, b \in \mathbb{R}$. Then the distance between a and b is defined by:

$$d_{\mathbb{R}}(a, b) = |a - b|.$$

Theorem 1.3. Properties of absolute value are described by the following:

- (a) $|ab| = |a| |b|$ for all $a, b \in \mathbb{R}$.
- (b) $|a|^2 = a^2$ for all $a \in \mathbb{R}$.
- (c) If $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$.
- (d) $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

Proof. We shall prove the properties described above:

- (a). We shall use the algebraic characterization of absolute value to prove this as follows. Since $|a| = \sqrt{a^2}$, we have:

$$|ab| = \sqrt{a^2 b^2} = \sqrt{a^2} \sqrt{b^2} = |a| |b|$$

as desired.

- (b). Since $a^2 \geq 0$, we have $a^2 = |a^2| = |aa| = |a||a|$. Also, by the algebraic characterization of absolute value we have: $|a|^2 = \sqrt{a^2}^2 = a^2$, as desired.
- (c). If $|a| \leq c$, then we have both $a \leq c$ and $-a \leq c$. Indeed, if $a \geq 0$, then $a \leq c$ and if $a \leq 0$, then $-a \leq c$ as required. Consequently, $-a \leq c$ implies, $-c \leq a$, so that $-c \leq a \leq c$. Conversely, if $-c \leq a \leq c$, then we have both $a \leq c$ and $-a \leq c$, so that $|a| \leq c$.
- (d). Let $c = |a|$, and use a similar argument as in the proof of (3.).

□

Triangle Inequality. If $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

Corollary 1.4. For all $a, b \in \mathbb{R}$, the following hold:

- (a). $||a| - |b|| \leq |a - b|$,

(b). $|a - b| \leq |a| + |b|$.

Proof. We prove the above respectively:

(a). Take $a := a - b + b$ and apply the triangle inequality as follows:

$$|a| = |(a - b) + b| \leq |a - b| + |b|$$

then by subtracting from $|b|$, we get

$$|a| - |b| \leq |a - b|.$$

Similarly, set $b := b - a + a$ and apply the triangle inequality as follows:

$$|b| = |(b - a) + a| \leq |b - a| + |a|$$

then by subtracting from $|a|$, we get

$$|b| - |a| \leq |b - a|.$$

Then, by virtue of **Theorem 1.3.(c)**, we have $||a| - |b|| \leq |a - b|$.

(b). Take $b := -b$ in the triangle inequality and recall that $|-b| = |b|$, thus we have

$$|a + (-b)| \leq |a| + |-b| = |a| + |b|$$

as desired.

□

Corollary 1.5. *Some properties of the distance function on \mathbb{R} . For all $x, y, z \in \mathbb{R}$, the following hold:*

- (a). **Positive Definiteness:** $d_{\mathbb{R}}(x, y) \geq 0$.
- (b). **Symmetry:** $d_{\mathbb{R}}(x, y) = d_{\mathbb{R}}(y, x)$.
- (c). **Triangle Inequality:** $d_{\mathbb{R}}(x, y) \leq d_{\mathbb{R}}(x, z) + d_{\mathbb{R}}(y, z)$.

Proof. We prove the above properties respectively:

(a). By definition, $d_{\mathbb{R}}(x, y) = |x - y|$, which by the algebraic characterization of absolute value is:

$$\sqrt{(x - y)^2}$$

from which we deduce that $(x - y)^2 \geq 0$, hence the square root of this quantity must be greater than or equal to zero, and equality holds if and only if $x = y$.

- (b). For $x, y \in \mathbb{R}$, we know that $d_{\mathbb{R}}(x, y) = |x - y|$, which by algebraic characterization gives us:

$$\begin{aligned}\sqrt{(x - y)^2} &= \sqrt{(y - x)^2} \\ &= |y - x| \\ &= d_{\mathbb{R}}(y, x).\end{aligned}$$

- (c). For $x, y, z \in \mathbb{R}$,

$$d_{\mathbb{R}}(x, y) = |(x - z) + (z - y)| \leq |x - z| + |z - y|$$

for which the right-handed side is equivalent to $d_{\mathbb{R}}(x, z) + d_{\mathbb{R}}(z, y)$, as desired.

□

Definition 1.6. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε -neighborhood of a is the set

$$V_{\varepsilon}(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\}.$$

Theorem 1.7. Let $a \in \mathbb{R}$. If x belongs to the neighborhood $V_{\varepsilon}(a)$ for every $\varepsilon > 0$, then $x = a$.

Proof. Suppose on the contrary that $x \neq a$. Then, for $\varepsilon_0 = |x - a| > 0$, we have by definition

$$|x - a| < \varepsilon_0 = |x - a|,$$

a contradiction. Therefore $x = a$ must be true.

□

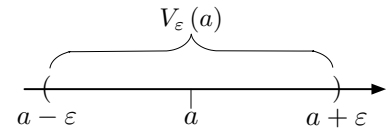


Figure 1.1: ε -neighborhood. For $a \in \mathbb{R}$, the statement $x \in V_{\varepsilon}(a)$ is equivalent to either of the statements:

$$-\varepsilon < x - a < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon.$$

2 The Completeness Property of the Set \mathbb{R} .

3 The Archimedean Property and Density of Rational Numbers in \mathbb{R} .

The Archimedean property states that for every real number $x \in \mathbb{R}$, there is a natural number $n_x \in \mathbb{N}$ such that $n_x > x$. The Archimedean property can be reformulated as follows:

Given $a, b \in \mathbb{R}$, where $a, b > 0$, there is $n \in \mathbb{N}$ such that $na > b$. Indeed $na > b$ if and only if $n > \frac{b}{a} = x \in \mathbb{R}$. So, by the Archimedean property such n exists.

If a represents the volume of a teaspoon and b represents the volume of a lake, then the above corollary of the Archimedean property states that given enough time $(n) \dots$, one can empty a lake with a teaspoon.

Archimedean Property. For all $x \in \mathbb{R}$, there exists $n_x \in \mathbb{N}$ such that $n_x > x$.

Proof. Suppose that the Archimedean Property is false for every real number, that is,

$$\forall x \in \mathbb{R}, n_x \leq x.$$

Then this implies that x is an upper bound for \mathbb{N} . By the completeness axiom, \mathbb{N} has a supremum in \mathbb{R} , that is,

$$\exists x_0 \in \mathbb{R}, \text{ such that } x_0 = \sup \mathbb{N}.$$

By Lemma 2.3.4, it follows that x_0 is a supremum of \mathbb{N} if and only if for every $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbb{N}$, such that $x_0 - \varepsilon < n_\varepsilon$. So, let $\varepsilon = 1$, so that $x_0 - 1 < n_\varepsilon$, which is equivalent to

$$x_0 < n_\varepsilon + 1,$$

but then $n_\varepsilon + 1$ is in \mathbb{N} , and it's greater than x_0 , thus a contradiction because x_0 is an upper bound of \mathbb{N} . \square

Density of Rational Numbers. If $x, y \in \mathbb{R}$ and $x < y$, then there is $\frac{m}{n} \in \mathbb{Q}$ such that

$$x < \frac{m}{n} < y.$$

Proof. There is no restriction in assuming that $0 < x < y$. Indeed if $x = 0$, just shift both x and y up one unit, that is, consider $\tilde{x} = x + 1$ and $\tilde{y} = y + 1$. Then clearly $0 < \tilde{x} < \tilde{y}$. If there exists $\tilde{r} \in \mathbb{Q}$ such that $\tilde{x} < \tilde{r} < \tilde{y}$, then $r = \tilde{r} - 1 \in \mathbb{Q}$, satisfying $x < r < y$ as needed. To be continued... \square

Corollary 3.1. (Density of Irrational Numbers) If $x, y \in \mathbb{R}$ and $x < y$, then there is $w \in \mathbb{R} \setminus \mathbb{Q}$, such that $x < w < y$.

Proof. Let $w_0 \in \mathbb{R} \setminus \mathbb{Q}$. By the density of rational numbers $\exists \frac{m}{n} \in \mathbb{Q}$, such that

$$\frac{x}{|w_0|} < \frac{m}{n} < \frac{y}{|w_0|}.$$

3. The Archimedean Property and Density of Rational Numbers in \mathbb{R} .

If $\frac{m}{n} = 0$, then pick another rational number between 0 and $\frac{y}{|w_0|}$ so that $\frac{m}{n} \neq 0$. Let $w = |w_0|\frac{m}{n}$, then $w \in \mathbb{R} \setminus \mathbb{Q}$ because the product of a rational number and an irrational number is irrational, and thus

$$x < w < y$$

as desired. □