

# PHYSICS

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## Homework 6 Solutions

### Problem 1

- (a) Find the normal frequencies for small oscillations of the double pendulum for arbitrary values of the masses and lengths.
- (b) Check that your answers are correct for the special case that  $m_1 = m_2$  and  $L_1 = L_2$ .
- (c) Discuss the limit that  $m_2 \rightarrow 0$

### Solution:

(a) The entire first part of this problem is actually worked out in Taylor as an example (pages 431-434), so I won't re-derive it here. Instead, I'll take over from where Taylor left off.

The equation of motion is  $\mathbf{M}\ddot{\phi} = -\mathbf{K}\phi$ , where  $\mathbf{M}$  and  $\mathbf{K}$  are given by equation 11.44 in Taylor:

$$\mathbf{M} = \begin{pmatrix} (m_1 + m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} (m_1 + m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{pmatrix}$$

The normal frequencies are found by proposing oscillatory solutions, writing them as the real part of an exponential, plugging them back into the equation of motion, and then solving  $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ , which only has solutions when  $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ . So we just have to find the values of  $\omega$  that satisfy that determinant equation. I did this entirely in Mathematica, because it's ugly and messy. I invite you to do the same. The answer is:

$$\omega^2 = \frac{g(L_1 + L_2)(m_1 + m_2) \pm g\sqrt{(m_1 + m_2)[(L_1 - L_2)^2m_1 + (L_1 + L_2)^2m_2]}}{2L_1L_2m_1}$$

which is just scary.

Notice we can re-write the equation for  $\mathbf{a}$  to make it more clear that it's just an eigenvalue equation:

$$\begin{aligned} (\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} &= 0 \\ \mathbf{K}\mathbf{a} &= \omega^2\mathbf{M}\mathbf{a} \\ \mathbf{M}^{-1}\mathbf{K}\mathbf{a} &= \omega^2\mathbf{a} \end{aligned}$$

so another way to find our values of  $\omega$  is to see that they are just the eigenvalues of that matrix.

(b) When  $m_1 = m_2$  and  $L_1 = L_2$ , the first term in the root cancels, and the rest is:

$$\omega^2 = \frac{g2L \cdot 2m \pm g\sqrt{2m^2 4L^2}}{2L^2m} = \frac{g2L \cdot 2m \pm g2m \cdot L\sqrt{2}}{2L^2m} = \frac{g}{L} (2 \pm \sqrt{2})$$

which is exactly equation 11.47 where Taylor describes that example.

(c) In the case when  $m_2 = 0$  we get:

$$\omega^2 = \frac{g(L_1 + L_2)m_1 \pm g\sqrt{m_1^2(L_1 - L_2)^2}}{2L_1L_2m_1} = \frac{g(L_1 + L_2 \pm (L_1 - L_2))}{2L_1L_2} = \frac{g}{L_{1,2}}$$

which is what we would expect: the pendulum has been reduced to a simple pendulum. The fact that we can get a result with  $L_1$  or with  $L_2$  simply stems from the fact that  $\omega$  is symmetric in the two lengths: it doesn't really matter where the heavier mass is. We would obviously choose  $L_1$  in our case, because the other one makes no sense.

## Problem 2

Two equal masses are constrained to move without friction, one on the positive  $x$  axis and one on the positive  $y$  axis. They are attached to two identical springs (force constant  $k$ ) whose other ends are attached to the origin. In addition, the two masses are connected to each other by a third spring of force constant  $k'$ . The springs are chosen so that the system is in equilibrium with all three springs relaxed. What are the normal frequencies? Find and describe the normal modes.

### Solution:

As usual, the complicated part about this problem is figuring out the potential energy. The issue here is the annoying  $k'$  spring, which is diagonal. Now, drawing springs is really hard, so just pay attention to my description.

Let  $L$  be the length of the  $k'$  spring when the other two are stretched a distance  $x$  and  $y$  respectively, and let  $\ell$  be the rest length of the horizontal springs. From basic trigonometry, the rest length of the  $k'$  spring is  $\sqrt{2}\ell$ . Now, the distance we're actually interested in is  $\Delta L = L - \sqrt{2}\ell$ , since this is what determines the force this spring exerts. In what follows, I'll assume  $x$  and  $y$  are small, so I'll only keep terms up to second order in them.

We can express  $\Delta L$  as:

$$\begin{aligned}\Delta L &= \sqrt{(\ell + x)^2 + (\ell + y)^2} - \sqrt{2}\ell \\ &= \sqrt{2\ell^2 + x^2 + y^2 + 2\ell x + 2\ell y} - \sqrt{2}\ell \\ &= \sqrt{2}\ell \sqrt{1 + \frac{x^2 + y^2}{2\ell^2} + \frac{x + y}{\ell}} - \sqrt{2}\ell \\ &\approx \sqrt{2}\ell \left( 1 + \frac{x^2 + y^2}{4\ell^2} + \frac{x + y}{2\ell} - 1 \right) \\ \Delta L &= \frac{\sqrt{2}}{2} \left( \frac{x^2 + y^2}{2\ell} + x + y \right)\end{aligned}$$

With that, our potential energy is

$$\begin{aligned}U &= \frac{1}{2}k(x^2 + y^2) + \frac{1}{2}k'\Delta L^2 \\ &= \frac{1}{2}k(x^2 + y^2) + \frac{1}{4}k' \left( \frac{x^2 + y^2}{2\ell} + x + y \right)^2 \\ &\approx \frac{1}{2}k(x^2 + y^2) + \frac{1}{4}k'(x^2 + y^2 + 2xy)\end{aligned}$$

where in the last step I got rid of all terms of order higher than 2.

With that potential energy, our Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2) - \frac{1}{4}k'(x^2 + y^2 + 2xy)$$

so our equations of motion are

$$\begin{aligned} m\ddot{x} &= -x\left(k + \frac{1}{2}k'\right) - \frac{1}{2}k'y \\ m\ddot{y} &= -y\left(k + \frac{1}{2}k'\right) - \frac{1}{2}k'x \end{aligned}$$

or  $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$ , where

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} k + \frac{1}{2}k' & \frac{1}{2}k' \\ \frac{1}{2}k' & k + \frac{1}{2}k' \end{pmatrix}$$

Now we do our usual thing: assume oscillatory solutions and re-write the matrix equation in terms of that, which will lead us to require that  $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ :

$$\det \begin{pmatrix} k + \frac{k'}{2} - \omega^2 m & \frac{k'}{2} \\ \frac{k'}{2} & k + \frac{k'}{2} - \omega^2 m \end{pmatrix} = \left(k + \frac{k'}{2} - \omega^2 m\right)^2 - \frac{k'^2}{4} = 0$$

Working on the equation:

$$k^2 + kk' + \omega^4 m^2 - 2\omega^2 m \left(k + \frac{k'}{2}\right) = 0$$

which gives us

$$\omega_1 = \pm \sqrt{\frac{k}{m}}, \quad \omega_2 = \pm \sqrt{\frac{k+k'}{m}}$$

This is already giving us a hint about the normal modes. In the first one, the diagonal spring plays no role, so we expect it isn't stretched. Let's see...

Using  $\omega_1$  first, our eigenvector equation becomes:

$$\mathbf{K} - \omega_1^2 \mathbf{M} = \frac{k'}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{k'}{2} \begin{pmatrix} a_1 + a_2 \\ a_1 + a_2 \end{pmatrix} = 0$$

This means that in this mode  $a_1 = -a_2$ . That is, the horizontal springs are oscillating exactly out of phase and with the same amplitude. As we expected, this means that the diagonal spring is always kept at the same length, so it plays no role in the equations of motion.

Using  $\omega_2$  we get:

$$\mathbf{K} - \omega_2^2 \mathbf{M} = \frac{k'}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{k'}{2} \begin{pmatrix} -a_1 + a_2 \\ a_1 - a_2 \end{pmatrix} = 0$$

This means that  $a_1 = a_2$ . In this case, the springs are oscillating perfectly in phase and with the same amplitude, thus stretching and compressing the diagonal spring, so its spring constant plays a role in the frequency of oscillations.

### Problem 3

A bead of mass  $m$  is threaded on a frictionless circular wire hoop of radius  $R$  and mass  $m$  (same mass). The hoop is suspended at the point  $A$  and it's free to swing in its own vertical plane. Using the angles  $\phi_1$  and  $\phi_2$  as generalized coordinates, solve for the normal frequencies of small oscillations, and find and describe the motion in the corresponding normal modes.

#### Solution:

I actually find Taylor's suggestions for angles unnecessarily complicated, so I'm using a slightly modified version described in the figure. The only difference is that the second angle is with respect to the vertical, not the diameter.

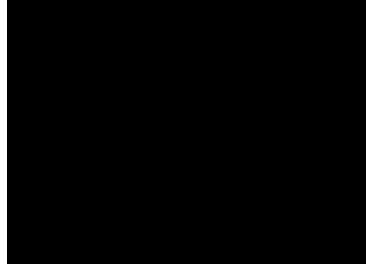


Figure 1: The variables used in Problem 3

Now, with those angles, the coordinates of the mass  $m$  are given by:

$$\begin{aligned}x_m &= R(\sin \phi_2 + \sin \phi_1) \\y_m &= -R(\cos \phi_2 + \cos \phi_1)\end{aligned}$$

and its kinetic energy is given by

$$\begin{aligned}T_m &= \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) \\&= \frac{1}{2}mR^2 \left[ (\dot{\phi}_1 \cos \phi_1 + \dot{\phi}_2 \cos \phi_2)^2 + (\dot{\phi}_1 \sin \phi_1 + \dot{\phi}_2 \sin \phi_2)^2 \right] \\&= \frac{1}{2}mR^2 \left[ \dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right] \\&\approx \frac{1}{2}mR^2(\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1 \dot{\phi}_2)\end{aligned}$$

where in the last line I used the small angle approximation up to second order.

The kinetic energy of the hoop is only rotational. The only "tricky" part is finding the right moment of inertia, but that's easy, because this is a 2D problem, so  $I$  is basically a scalar. We just have to move it from the center to point  $A$  by using the parallel axes theorem:  $I = I_{\text{cm}} + mR^2 = mR^2 + mR^2 = 2mR^2$ , so now our total kinetic energy is:

$$T_{\text{tot}} = mR^2 \dot{\phi}_1^2 + \frac{1}{2}mR^2(\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1 \dot{\phi}_2)$$

To find the potential energy, all we need is the  $y$  coordinate of the center of the hoop. This is given by  $y_{\text{hoop}} = -R \cos \phi_1$ , so our potential energy is

$$U_{\text{tot}} = -2mgR \cos \phi_1 - mgR \cos \phi_2 \approx \frac{1}{2}mgR(2\phi_1^2 + \phi_2^2) + U_0$$

where, again, I approximated up to second order and put all the constants in  $U_0$  since they won't show up in the equations of motion.

After all the algebra with the Lagrangian, our equations of motion turn out to be:

$$\begin{aligned} 3\ddot{\phi}_1 + \ddot{\phi}_2 &= -\frac{2g}{R}\phi_1 \\ \ddot{\phi}_1 + \ddot{\phi}_2 &= -\frac{g}{R}\phi_2 \end{aligned}$$

which we can express in matrix notation by defining

$$\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{K} = \frac{g}{R} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

so now our equation is simply  $\mathbf{M}\ddot{\boldsymbol{\phi}} = -\mathbf{K}\boldsymbol{\phi}$ . The rest is the same old spiel: guess an exponential solution with frequency  $\omega$ , plug it back in, write in a form that necessitates a zero determinant, solve for  $\omega$ . So our equation for  $\omega$  is lkjahsdkjhwkegkdfaksdjkagsdjhga,jhsdga-jkgsdkjahskdjhakjsdhkaj aksjdhkajhsdkjha aksjdhkahjsdk asdkjhaksjda skjx kajshdkjbasc e: